ECE 604, Lecture 5

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Contents

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1 Derivation of Biot-Savart Law

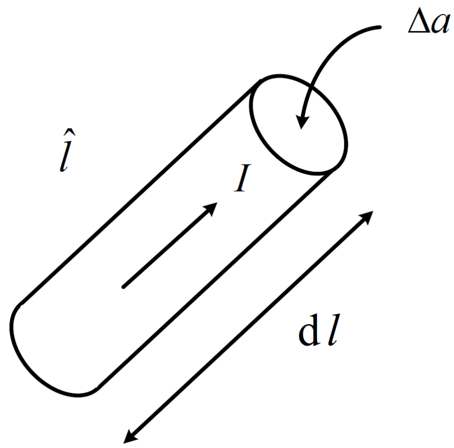

Figure 1:

Biot-Savart law, like Ampere's law was experimentally determined in around 1820. This is the cumulative work of Ampere, Oersted, Biot, and Savart. Now, we have the mathematical tool to derive this law from Ampere's law and Gauss's law.

From Gauss' law and Ampere's law, we have derived that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{R} dV' \tag{1.1}$$

When the current element is small, and is carried by a wire of cross sectional area Δa as shown in Figure 1, we can approximate the integrand as

$$\mathbf{J}(\mathbf{r}')dV' \approx \mathbf{J}(\mathbf{r}')\Delta V' = \underbrace{(\Delta a)\Delta l}_{\Delta V} \underbrace{\hat{l} I/\Delta a}_{\mathbf{J}(\mathbf{r}')} \tag{1.2}$$

In the above, $\Delta V = (\Delta a)\Delta l$ and $\hat{l} I/\Delta a = \mathbf{J}(\mathbf{r}')$ since $\mathbf{J}$ has the unit of amperes/m^2. Here, $\hat{l}$ is a unit vector pointing in the direction of the current flow. Hence, we can let

$$\mathbf{J}(\mathbf{r}')dV' \approx I\Delta\mathbf{l} \tag{1.3}$$

where the vector $\Delta\mathbf{l} = \Delta l\hat{l}$. Therefore, the incremental vector potential due to an incremental current is

$$\Delta\mathbf{A}(\mathbf{r}) \approx \frac{\mu}{4\pi}\left(\frac{\mathbf{J}(\mathbf{r}')\Delta V'}{R}\right) = \frac{\mu}{4\pi}\frac{I\Delta\mathbf{l}'}{R} \tag{1.4}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we derive that the incremental $\mathbf{B}$ flux is

$$\Delta\mathbf{B} = \nabla \times \Delta\mathbf{A}(\mathbf{r}) \cong \frac{\mu I}{4\pi}\nabla \times \frac{\Delta\mathbf{l}'}{R} = \frac{-\mu I}{4\pi}\Delta\mathbf{l}' \times \nabla\frac{1}{R} \tag{1.5}$$

where we have made use of the fact that $\nabla \times \mathbf{a}f(\mathbf{r}) = -\mathbf{a} \times \nabla f(\mathbf{r})$ when $\mathbf{a}$ is a constant vector. The above can be simplified further by making use of the fact that

$$\nabla\frac{1}{R} = -\frac{1}{R^2}\hat{R} \tag{1.6}$$

where $\hat{R}$ is a unit vector pointing in the $\mathbf{r}-\mathbf{r}'$ direction. We have also made use of the fact that $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Consequently, assuming that the incremental length becomes very small, or $\Delta\mathbf{l} \to \mathbf{dl}$, we have, after using (1.6) in (1.5), that

$$\mathbf{dB} = \frac{\mu I}{4\pi}\mathbf{dl}' \times \frac{1}{R^2}\hat{R} \tag{1.7}$$

$$= \frac{\mu I \mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{1.8}$$

Since $\mathbf{B} = \mu\mathbf{H}$, we have

$$\mathbf{dH} = \frac{I\mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{1.9}$$

or

$$\mathbf{H}(\mathbf{r}) = \int \frac{I(\mathbf{r}')\mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{1.10}$$

which is Biot-Savart law

2 Boundary Conditions–Conductive Media Case

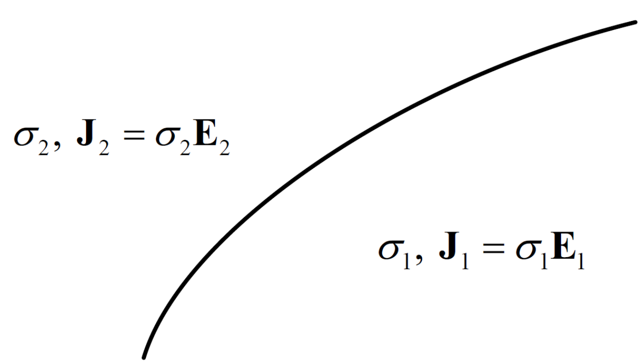

Figure 2:

From the current continuity equation, one gets

$$\nabla \cdot \mathbf{J} = -\frac{\partial \varrho}{\partial t} \tag{2.1}$$

If the right-hand side is everywhere finite, it will not induce a jump discontinuity in the current. Moreover, it is zero for static case. Hence, just like the Gauss's law case, the above implies that the normal component of the current J_n is continuous, or that $J_{1n} = J_{2n}$. In other words,

$$\hat{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) = 0 \tag{2.2}$$

Hence, using $\mathbf{J} = \sigma \mathbf{E}$, we have

$$\sigma_2 E_{2n} - \sigma_1 E_{1n} = 0 \tag{2.3}$$

But Gauss's law implies the boundary condition that

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \varrho_s \tag{2.4}$$

Hence, surface charge density or charge accumulation is necessary, unless $\sigma_2/\sigma_1 = \varepsilon_2/\varepsilon_1$.

2.1 Electric Field Inside a Conductor

The electric field inside a perfect conductor has to be zero. If medium 1 is a perfect conductor, then $\sigma \to \infty$ but $\mathbf{J}_1 = \sigma \mathbf{E}_1$. An infinitesimal small $\mathbf{E}_1$ will give rise to an infinite current $\mathbf{J}_1$. To avoid this ludicrous situation, thus $\mathbf{E}_1 = 0$. This implies that $\mathbf{D}_1 = 0$ as well.

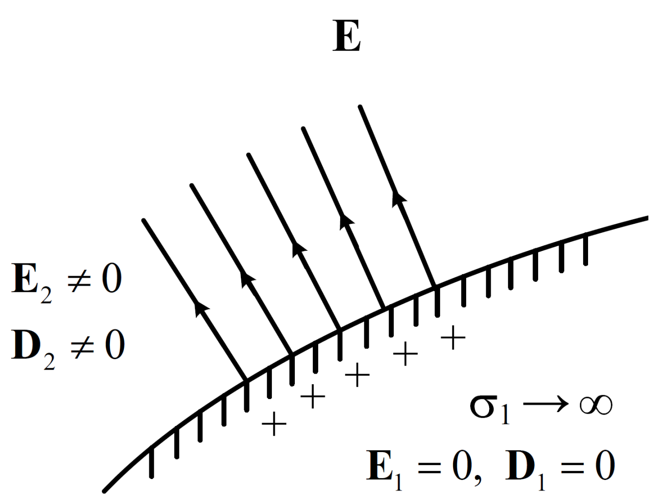

Figure 3:

Since tangential **E** is continuous, from Faraday's law, it is still true that

$$E_{2t} = E_{1t} = 0 \tag{2.5}$$

But since

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{2.6}$$

and that $\mathbf{D}_1 = 0$, then

$$\hat{n} \cdot \mathbf{D}_2 = \varrho_s \tag{2.7}$$

In other words, normal $\mathbf{D}_2 \neq 0$, tangential $\mathbf{E}_2 = 0$. The sketch of the electric field in the vicinity of a perfect conducting surface is shown in Figure 3.

The above argument for zero electric field inside a perfect conductor is true for electrodynamic problems. However, one does not need the above argument regarding the shielding of the static electric field from a conducting region. In the situation of the two conducting objects example below, as long as the electric fields are non-zero in the objects, currents will keep flowing. They flow until the charges in the two objects orient themselves so that electric current cannot flow anymore. This happens when the charges produce internal fields that cancel each other giving rise to zero field inside the two objects. Faraday's law still applies which means that tangental **E** field has to be continuous. Therefore, the boundary condition that the fields have to be normal to the conducting object surface is still true for elecrostatics. A sketch of the electric field between two conducting spheres is show in Figure 4.

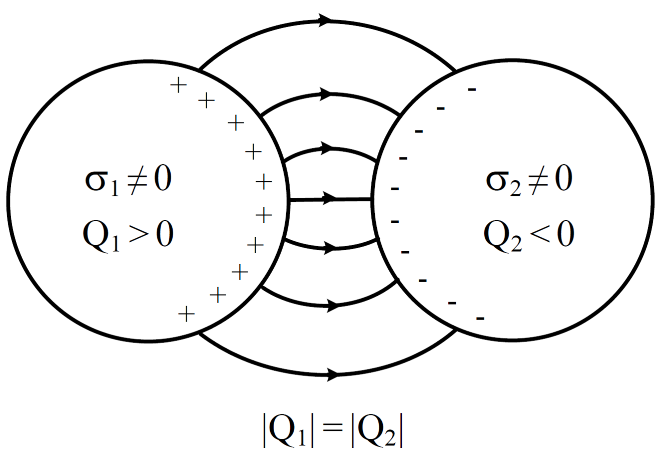

Figure 4:

2.2 Magnetic Field Inside a Conductor

We have seen that for a finite conductor, as long as $\sigma \neq 0$, the charges will re-orient themselves until the electric field is expelled from the conductor; otherwise, the current will keep flowing. But there are no magnetic charges nor magnetic conductors in this world. So this physical phenomenon does not happen for magnetic field: in other words, magnetic field cannot be expelled from an electric conductor. However, a magnetic field is expelled from a perfect conductor or a superconductor. You can only fully understand this physical phenomenon if we study Maxwell's equations in their full glory or in their time-varying form.

In a perfect conductor where $\sigma \to \infty$, it is unstable for the magnetic field $\mathbf{B}$ to be nonzero. As time varying magnetic field gives rise to an electric field by the time-varying form of Faraday's law, a small time variation of the $\mathbf{B}$ field will give rise to infinite current flow in a perfect conductor. Therefore to avoid this ludicrous situation, and to be stable, $\mathbf{B} = 0$ in a perfect conductor or a superconductor.

So if medium 1 is a perfect electric conductor, then $\mathbf{B}_1 = \mathbf{H}_1 = 0$. The boundary conditions from Ampere's law and Gauss' law for magnetic flux give rise to

$$\hat{n} \times \mathbf{H}_2 = \mathbf{J}_s \tag{2.8}$$

which is the jump condition for the magnetic field. The magnetic flux $\mathbf{B}$ is expelled from the perfect conductor, and there is no normal component of the $\mathbf{B}$ field as there cannot be magnetic charges. Therefore, the boundary condition becomes

$$\hat{n} \cdot \mathbf{B}_2 = 0 \tag{2.9}$$

The $\mathbf{B}$ field in the vicinity of a conductor surface is as shown in Figure 5.

When a superconductor cube is placed next to a static magnetic field near a permanent magnet, eddy current will be induced on the superconductor. The eddy current will expel the static magnetic field from the permanent magnet, or it will produce a magnetic dipole on the superconducting cube that repels the static magnetic field. This causes the superconducting cube to levitate on the static magnetic field.

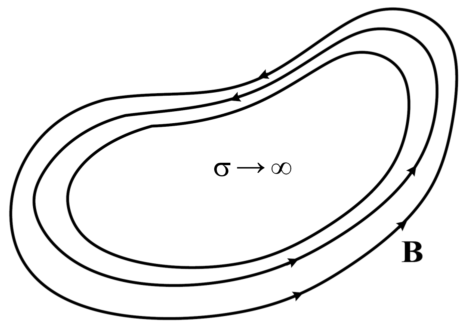

Figure 5:

3 Instantaneous Poynting's Theorem

Before we proceed further with studying energy and power, it is habitual to add fictitious magnetic current $\mathbf{M}$ and fictitious magnetic charge ϱ_m to Maxwell's equations to make them mathematically symmetrical. To this end, we have

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} \tag{3.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{3.2}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = \varrho_m \tag{3.4}$$

Consider the first two of Maxwell's equations where fictitious magnetic current is included and that the medium is isotropic such that $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$. Next, we need to consider only the first two equations since in electrodynamics, by invoking charge conservation, the third and the fourth equations are derivable from the first two. They are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M}_i = -\mu\frac{\partial \mathbf{H}}{\partial t} - \mathbf{M}_i \tag{3.5}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \varepsilon\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_i + \sigma\mathbf{E} \tag{3.6}$$

where $\mathbf{M}_i$ and $\mathbf{J}_i$ are impressed current sources. They are sources that are impressed into the system, and they cannot be changed by their interaction with the environment.

Also, for a conductive medium, a conduction current or induced current flows in addition to impressed current. Here, $\mathbf{J} = \sigma\mathbf{E}$ is the induced current source.

Moreover, $\mathbf{J} = \sigma\mathbf{E}$ is similar to ohm's law. We can show from (3.5) and (3.6) that

$$\mathbf{H} \cdot \nabla \times \mathbf{E} = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \mathbf{H} \cdot \mathbf{M}_i \tag{3.7}$$

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot \mathbf{J}_i + \sigma \mathbf{E} \cdot \mathbf{E} \tag{3.8}$$

Using the identity, which is the same as the product rule for derivatives, we have[1]

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{3.9}$$

Therefore, from (3.7), (3.8), and (3.9) we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i \right) \tag{3.10}$$

The physical meaning of the above is more lucid if we first consider $\sigma = 0$, and $\mathbf{M}_i = \mathbf{J}_i = 0$, or the absence of conductive loss and the impressed current sources. Then the above becomes

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \right) \tag{3.11}$$

Rewriting each term on the right-hand side of the above, we have

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} \mathbf{H} \cdot \mathbf{H} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\mathbf{H}|^2 \right) = \frac{\partial}{\partial t} W_m \tag{3.12}$$

$$\varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \varepsilon \frac{\partial}{\partial t} \mathbf{E} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon |\mathbf{E}|^2 \right) = \frac{\partial}{\partial t} W_e \tag{3.13}$$

Then (3.11) becomes

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} (W_m + W_e) \tag{3.14}$$

where

$$W_m = \frac{1}{2} \mu |\mathbf{H}|^2, \qquad W_e = \frac{1}{2} \varepsilon |\mathbf{E}|^2 \tag{3.15}$$

Equation (3.14) is reminiscent of the current continuity equation, namely,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \varrho}{\partial t} \tag{3.16}$$

which is a statement of charge conservation. In other words, time variation of current density at a point is due to charge density flow into or out of the point.

[1] The identity that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ is useful for the derivation.

Hence, $\mathbf{E} \times \mathbf{H}$ has the meaning of power density, and W_m and W_e are the energy density stored in the magnetic field and electric field, respectively. In fact, one can show that $\mathbf{E} \times \mathbf{H}$ has the unit of V m^{-1} times A m^{-1} which is W m^{-2}, where V is volt, A is ampere, and W is watt, which is joule s^{-1}. Hence, it has the unit of power density.

Similarly, $W_m = \frac{1}{2}\mu|\mathbf{H}|^2$ where μ has unit of H m^{-1}. Hence, W_m has the unit of H m^{-1} times A^2 m^{-2} = J m^{-3}, where H is henry, A is ampere, and J is joule. Therefore, it has the unit of energy density. We can also ascertain the unit of $\frac{1}{2}\mu|\mathbf{H}|^2$ easily by noticing that the energy stored in an inductor is $\frac{1}{2}LI^2$ which is in terms of joules, and is due to henry times A^2.

Also $W_e = \frac{1}{2}\varepsilon|\mathbf{E}|^2$ where ε has the unit of F m^{-1}. Hence, W_e has the unit of F m^{-1} times V^2 m^{-2} = J m^{-3} where F is farad, V is voltage, and J is joule, which is energy density again. We can also ascertain the unit of $\frac{1}{2}\varepsilon|\mathbf{E}|^2$ easily by noticing that the energy stored in a capacitor is $\frac{1}{2}CV^2$ which has the unit of joules, and is due to farad times V^2.

The vector quantity

$$\mathbf{S}_p = \mathbf{E} \times \mathbf{H} \tag{3.17}$$

is called the Poynting's vector, and (3.14) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t} W_t \tag{3.18}$$

where $W_t = W_e + W_m$ is the total energy density stored. The above is similar to the current continuity equation mentioned above. Analogous to that current density is charge density flow, power density is energy density flow.

Now, if we let $\sigma \neq 0$, then the term to be included is then $\sigma \mathbf{E} \cdot \mathbf{E} = \sigma|\mathbf{E}|^2$ which has the unit of S m^{-1} times V^2 m^{-2}, or W m^{-3} where S is siemens. We gather this unit by noticing that $\frac{1}{2}\frac{V^2}{R}$ is the power dissipated in a resistor of R ohms with a unit of watts. The reciprocal unit of ohms, which used to be mhos is now siemens. With $\sigma \neq 0$, (3.18) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t} W_t - \sigma|\mathbf{E}|^2 = -\frac{\partial}{\partial t} W_e - P_d \tag{3.19}$$

Here, $\nabla \cdot \mathbf{S}_p$ has physical meaning of power density oozing out from a point, and $-P_d = -\sigma|\mathbf{E}|^2$ has the physical meaning of power density dissipated (siphoned) at a point by the conductive loss in the medium which is proportional to $-\sigma|\mathbf{E}|^2$.

Now if we set $\mathbf{J}_i$ and $\mathbf{M}_i$ to be nonzero, (3.19) is augmented by the last two terms in (3.10), or

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t} W_t - P_d - \mathbf{H} \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i \tag{3.20}$$

The last two terms can be interpreted as the power density supplied by the impressed currents $\mathbf{M}_i$ and $\mathbf{J}_i$. Hence, (3.20) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t} W_t - P_d + P_s \tag{3.21}$$

where

$$P_s = -\mathbf{H} \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i \tag{3.22}$$

where P_s is the power supplied by the impressed current sources. These terms are positive if $\mathbf{H}$ and $\mathbf{M}_i$ have opposite signs, or if $\mathbf{E}$ and $\mathbf{J}_i$ have opposite signs. The last terms reminds us of what happens in a negative resistance device or a battery. In a battery, positive charges move from a region of lower potential to a region of higher potential (see Figure 6). The positive charges move from one end of a battery to the other end of the battery. Hence, they are doing an "uphill climb" due to chemical processes within the battery.

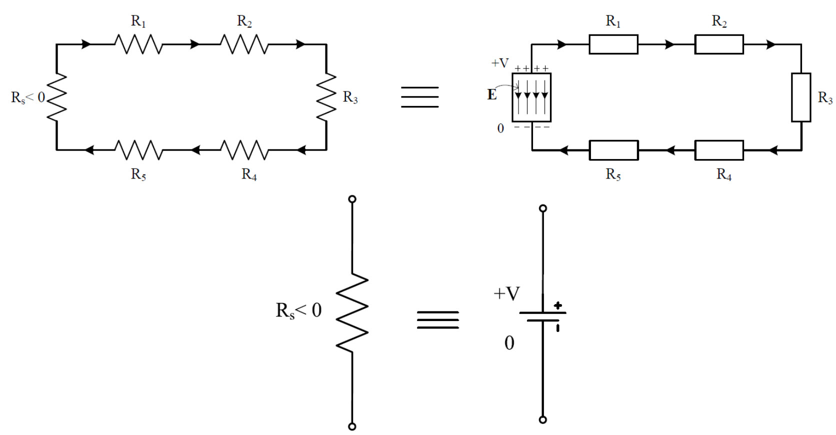

Figure 6:

In the above, one can easily work out that P_s has the unit of W m^{-3} which is power supplied density. One can also choose to rewrite (3.21) in integral form by integrating it over a volume V and invoking the divergence theorem yielding

$$\int_S d\mathbf{S} \cdot \mathbf{S}_p = -\frac{d}{dt}\int_V W_t dV - \int_V P_d dV + \int_V P_s dV \tag{3.23}$$

The left-hand side is

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}) \tag{3.24}$$

which represents the power flowing out of the surface S.